Accurate Amplitude Distribution Analyzer Combining Analog and Digital Logic

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A new precision analyzer yields digital readout of probability or probability density for random waveforms at low audio frequencies. A preamplifier-limiter conveniently increases a half-volt slicing interval to 20 v or more, and a sample-hold circuit permits the slicer to work slowly and accurately. The use of analog computer techniques permits convenient assembly of such instruments from inexpensive commercial plug-in amplifiers and decimal counters. Some statistical theory is also presented.

1. INTRODUCTION. SLICER PRINCIPLE

MANY studies of random signals, such as speech, receiver and control system noise, atmospheric turbulence, and sea water level and pressure require measurements approximating the first-order amplitude distribution of a random signal $x(t)$. In the case of stationary processes, the ensemble probability

$$P=\text{Prob}[X-(\Delta X/2)<x(t)\leq X+(\Delta X/2)] \qquad (1)$$

is independent of the time t and can in many cases be estimated by a suitable time average of the random variable,

$$y(t)=y[x(t)]=\begin{cases}1 \text{ if } x-(\Delta X/2)<x(t)\leq X+(\Delta X/2)\\ 0 \text{ otherwise}\end{cases} \qquad (2)$$

whose expected value $E\{y\}$ is necessarily equal to the desired probability (1). A function generator which produces the output (2) from a given input signal $x(t)$ will be called a (double) slicer circuit; Fig. 1 shows its transfer characteristic as well as typical input and output voltages as functions of time. If ΔX is made small (0.1 to 1% of the signal range), then $P/\Delta X$, and thus $E\{y\}/\Delta X$ approximates the probability density $\varphi(X)$ of $x(t)$.

2. TIME AVERAGES OF $y(t)$ AS PROBABILITY ESTIMATES

To estimate $P=E\{y\}$ one may take the average (periodic sample average)

$$[y]_{\text{av}}=\frac{1}{n}\sum_{k=1}^{n} y(k\Delta t)=\frac{1}{T}\sum_{k=1}^{n} y(k\Delta t)\Delta t \qquad (3)$$

of n sample values $y(k\Delta t)$ obtained at successive periodic sampling times $k\Delta t(k=1,2,\cdots,n)$; $T=n\Delta t$ will be called the integration time.

It is important to realize that the finite-time average (3) is a random variable whose value will fluctuate from sample to sample. Since $E\{[y]_{\text{av}}\}=E\{y\}=P$, $[y]_{\text{av}}$ is an unbiased estimate[1] of the desired probability P; but the variance of $[y]_{\text{av}}$, derived in appendix A, expresses a statistical error over and above any errors in the apparatus as such. This variance can be controlled in two practically important cases.

1. If the integration time $T=n\Delta t$ is held fixed while the sampling rate $1/\Delta t$ is increased, the sum (3) approximates the continuous finite-time average

$$\langle y\rangle_T=\frac{1}{T}\int_0^T y(t)dt, \qquad (4)$$

whose variance is given in Appendix A.

2. If successive samples $y(k\Delta t)$ are statistically independent (in practice this requires a relatively low sampling

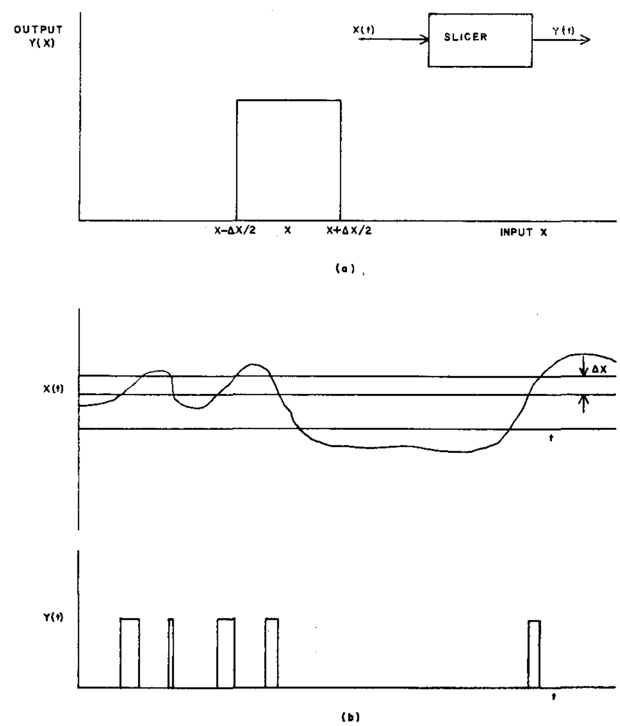


FIG. 1. Slicer operation.

[1] A function $y(x_1,x_2,\cdots,x_n)$ of n sample values x_k is an unbiased estimate of a theoretical parameter η if $E\{y\}=\eta$. y is a consistent estimate of η if Prob $[|y(x_1,x_2,\cdots,x_n)-\eta|>\epsilon]\to 0$ for every positive ϵ as the sample size n increases. For a brief review of the theory of estimation, see e.g., G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers* (McGraw-Hill Book Company, Inc., New York, 1961).

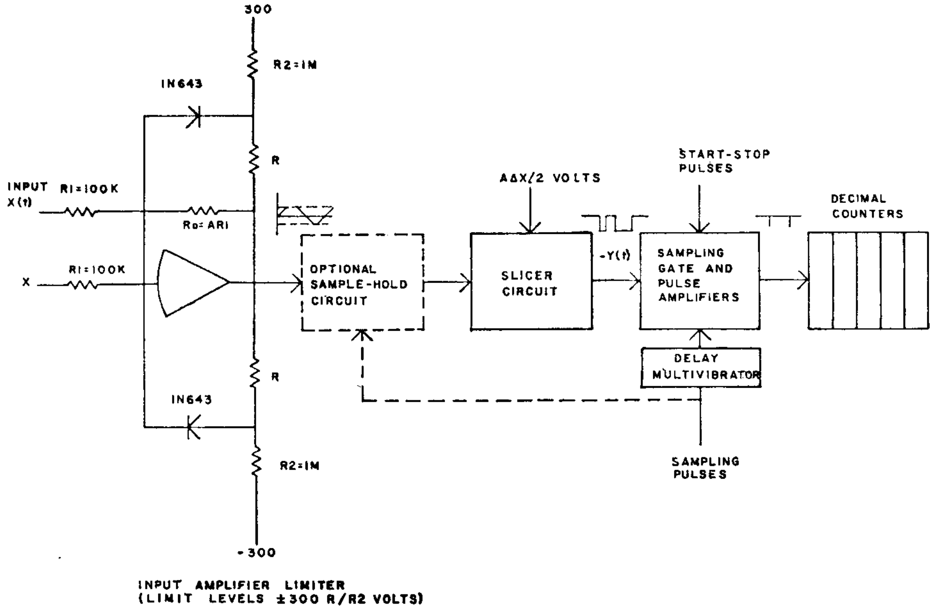


FIG. 2. Amplitude distribution analyzer.

rate $1/\Delta t$ not harmonically related to any frequencies of possible periodic signal components), then $n[y]_{av}$ has the familiar binomial distribution. In this case, the statistic (3) is a consistent, unbiased estimate of P with variance $P(1-P)/n$; one can reduce this variance at will by increasing the sample size n.

These two situations generate two separate methods for measuring (estimating) P with practical equipment.

3. AMPLITUDE DISTRIBUTION ANALYZER SYSTEMS: MODES OF OPERATION

In the block diagram of Fig. 2, the slicer center level X is first subtracted from the input $x(t)$ to make the slicing interval symmetric about zero voltage. The input amplifier limiter then amplifies (or attenuates) the input difference $x(t)-X$ to produce an output $A[x(t)-X]$ such that $\pm A\Delta X$ corresponds to, say, 90% of the output dynamic range (±50 v). The limiter diodes shown simply clip any portion of $A[x(t)-X]$ which exceeds 50 v in absolute value. This method multiplies the slicing level accuracy by A and is particularly effective if small slicing intervals ΔX are used, as in probability density measurements.

The amplified and limited version of $x(t)-X$ is now applied to the slicer circuit, either directly or through a sample-hold circuit. The slicer output voltage will be either on or off (+20 v or zero), so that samples $y(k\Delta t)$ obtained with a simple coincidence gate represent either 1 or 0 and can be counted with ordinary decimal counting units for a preset number n of sampling pulses. Alternatively, the slicer output, sampled or not, may be integrated[2] for a preset time T. With sampling frequencies below 1 Mc, the direct reading counters are usually substantially more accurate as well as more convenient than integrators.

The amplitude distribution analyzer system of Fig. 2 permits two distinct modes of operation, corresponding to the two types of sampling enumerated earlier:

1. "Fast" or "continuous type" sampling of stationary random voltages at rates exceeding twice the maximum frequency of interest in the signal; the resulting sample averages $[y]_{av}$ approximate continuous finite-time averages $\langle y\rangle_T$. For this type of operation there is no reason for preceding the slicer circuit with a sample-hold device.
2. "Slow" or "random" sampling of stationary random voltages at rates considerably lower than the maximum signal frequency. In this case a sample-hold circuit inserted ahead of the slicer permits the slicer to operate at the relatively low sampling frequency with greatly improved accuracy; only the input amplifier and the sample-hold circuit need to follow the signal itself.

4. ANALYSIS OF NONSTATIONARY RANDOM PROCESSES. ESTIMATION OF ENSEMBLE STATISTICS WITH A REPETITIVE ANALOG COMPUTER

A third mode of operation—and this will be the principal mode of operation in our Laboratory—applies to nonstationary as well as to stationary random processes and employs the analyzer to estimate values of probabilities (1) at a time $t=t_1$ from a sample of n sample functions ${}^kx(t)$. One presents the n sample functions of a given stationary or nonstationary random process to the analyzer, which samples each function t_1 seconds after it starts. In

[2] W. F. Caldwell, G. A. Korn, V. R. Latorre, and G. R. Peterson, Trans. IRE on Electronic Computers **EC-9**, 252 (1959).

this mode of operation, the analyzer yields estimates

$$[y]_{\mathrm{av}}=\frac{1}{n}\sum_{k=1}^{n}{}^{k}y(t_1) \tag{5}$$

of the ensemble average

$$E\{y(t_1)\}=\mathrm{Prob}[X-(\Delta X/2)<x(t_1)\leq X+(\Delta X/2)]. \tag{6}$$

In this connection, the usual sampling rates (5 to 100 cps) are again substantially lower than the maximum signal frequency, so that the use of the sample-hold circuit will improve the accuracy.

The sample functions could well be tape records of experimental data. The main intended application, however, is the statistical analysis of successive runs of a repetitive differential analyzer supplied with random initial conditions, parameters, and/or forcing functions from suitable random noise generators.

Differential analyzer records recur at repetition rates between 5 and 100 cps and may contain frequencies as high as 5 to 10 kc. In this connection the sample-hold circuit indicated in Fig. 2 permits the slicer circuit to operate on the low-frequency sample-hold output for more accurate slicer operation. The sample-hold circuit itself must be able to follow the high frequency input signal.

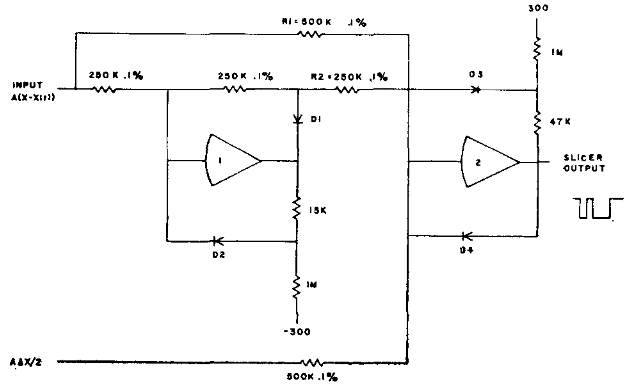
5. SLICER CIRCUIT

The precision dual slicer circuit[2] of Fig. 3(a) comprises a sensitive chopper stabilized operational-amplifier comparator[3] [amplifier 2 in Fig. 3(a)] whose output voltage changes abruptly between +50 v and −50 v whenever the comparator input voltage

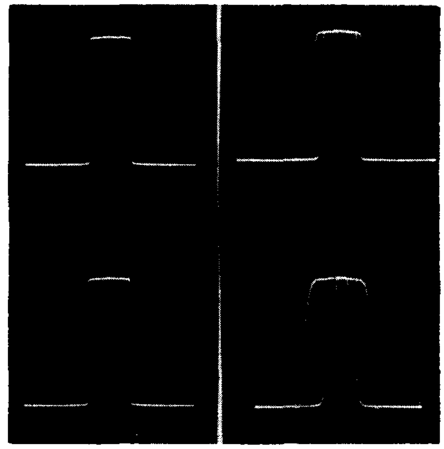
$$A(\Delta X/2)-A\,|X-x(t)|$$

changes sign. The comparator output is, thus, negative whenever $X-(\Delta X/2)<x(t)\leq X+(X/2)$. The slicing-width voltage $A(\Delta X/2)$ is set by a precision potentiometer. The absolute-value voltage $-A\,|X-x(t)|$ is added to the comparator input by an accurate analog computer circuit[3] involving a precision rectifier (amplifier 1)[3,4] together with resistors R_1 and R_2. Amplifier 1 is a chopper stabilized dc amplifier whose high gain cuts diode D_1 off decisively whenever the input voltage becomes negative; the input to resistor R_2 is then kept at zero by feedback through D_2. The resulting half-wave rectified input through R_2 combines with direct input through R_1 to yield the absolute-value input. Since a phase inverting pulse amplifier (V2b in Fig. 5) follows the slicer circuit, the slicer itself has negative output pulses; the combination of slicer and inverter has the transfer characteristic shown in Fig. 3(b) to provide positive gate pulses. Although reversal of diodes and bias voltages in Fig. 3(a) would yield a positive

[3] C. D. Morrill and R. V. Baum, Electronics **25**, 122 (1952).
[4] H. Koerner and G. A. Korn, Electronics **32**, No. 45, 66 (1959).



(a)



1 cps

100 cps

10 cps

1 kc

(b)

FIG. 3. (a) Dual slicer circuit. (b) Actual slicer transfer characteristics (output at plate of $V2_b$ in Fig. 5 vs slicer input) at 1, 10, and 100 cps, and 1 kc. Hysteresis is not noticeable at 10 cps (horizontal scale 5 v/cm; vertical scale 20 v/cm).

slicer output directly, the pulse amplifier permits one to use smaller slicer output pulses (−15 v) for better frequency response.

The static dc accuracy of the slicer circuit shown is better than 0.1 v. Figure 3(b) shows slicer transfer characteristics at 1, 10, 100 cps, and 1 kc. The comparator hysteresis at the highest typical operating frequency is less than 2 v, which is effectively divided by the preamplifier gain A (10 to 20) in probability density measurements.

6. SAMPLE-HOLD AND COINCIDENCE GATE CIRCUITS

The operational amplifier, sample-hold circuit shown in Fig. 4(a) acts like a phase inverting feedback amplifier when its six-diode switch[5] is on and can follow an 80-v peak-to-peak sinusoidal voltage at 1 kc. For even better frequency response, the input capacitor in Fig. 4(a) can be omitted, and a small follower amplifier (Philbrick K2-X) is used as a low impedance source to drive the gate; the

[5] J. Millman and H. Taub, *Pulse and Digital Circuits* (McGraw-Hill Book Company, Inc., New York, 1956).

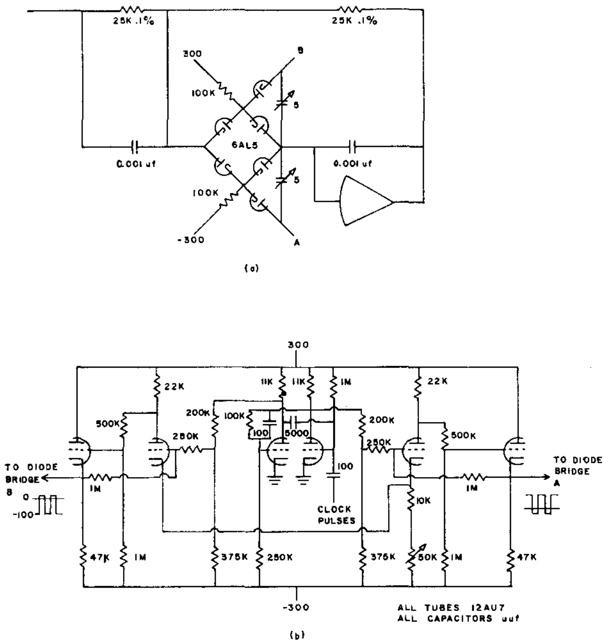


FIG. 4. Sample-hold circuit and sampling pulse amplifier.

resulting frequency response is 3 kc at 100-v peak-to-peak and 20 kc at 20 v peak-to-peak.[6]

When the electronic switch is turned off at the end of each 1 msec sampling pulse, the circuit acts like a feedback integrator without input except for leakage currents and holds its output voltage within 0.05 v until the sampling pulse reaches the coincidence gate (V4 in Fig. 5) through the delay multivibrator V3 in Fig. 5 (see also Fig. 2) 7 msec later. The sample-hold output then drifts slowly until the advent of the next sampling pulse. The sole purpose of the sample-hold circuit is to permit the slicer to settle comfortably during the 7 msec delay following each sampling pulse.

Referring now to Fig. 5, the coincidence gate V4 passes the delayed and differentiated sampling pulses from V2a to the readout counter if the slicer output is negative, i.e., if the signal $x(t)$ is between the slicing levels. The entire counting operation is started and stopped by the flip-flop V3, which is initially reset and applies a negative gate step to V4 when triggered by a stop pulse from a preset counter which counts the sampling pulses up to the preset sample size n.

The sampling pulse generator, comprising a monostable multivibrator with a push-pull amplifier and cathode follower, is shown in Fig. 4(b).

7. CONSTRUCTION AND RESULTS

The analyzer circuit uses Philbrick K2X/K2P plug-in dc amplifiers, except for the sample hold circuit, which uses a more powerful plugin amplifier developed at The University of Arizona.[7] Berkeley decimal counting units served as preset and readout counters. All semiconductor diodes used were type 1N643.

The analyzer identifies its slicing zone with a static dc accuracy better than 0.1 v, which is essentially determined by resistance tolerances.

In tests with low frequency signals (below 100 cps), the main difficulty was encountered in finding signal sources of sufficient accuracy. The analyzer clearly identified the diode function generator breakpoints in a Hewlett-Packard low frequency sine wave generator, as well as amplitude jitter in a phase shift oscillator. Finally, an accurate 10 cps triangle generator[4] was patched on an electronic analog computer. Test results, as shown in Fig. 6(a) indicated an accuracy of 0.1% of probability (probability measured in percent) for samples of $n = 10\,000$ and $\Delta X = 1$ v. The uniform probability distribution of a triangular waveform tends to conceal the effect of hysteresis errors, but the latter are minimized through the use of the preamplifier, as shown earlier.

For higher signal frequencies "slow" operation with statistically independent samples must be used, and some care is necessary to avoid sampling at rates too fast for statistical independence or harmonically related to periodic signal components. "Slow" or "random" sampling of a 1 kc triangular waveform at 100 samples/sec with

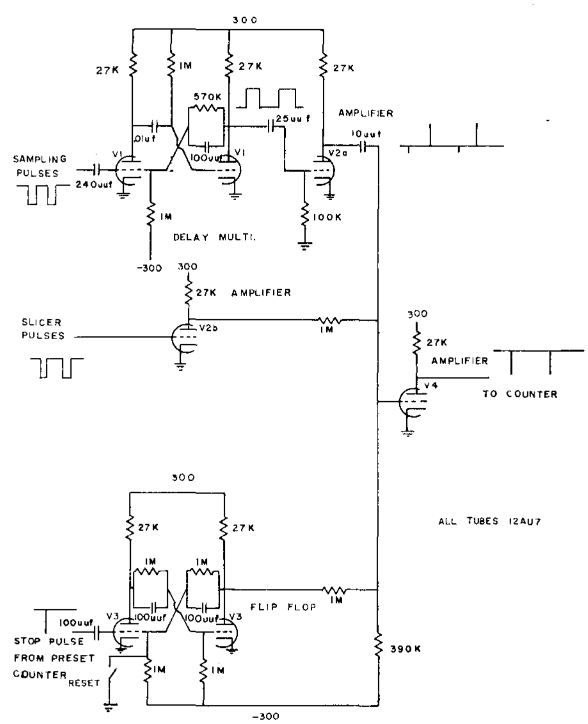


FIG. 5. Coincidence gate circuits sample the slicer output and stop the sample.

[6] T. A. Brubaker, ACL Memo. No. 22, Electrical Engineering Department, The University of Arizona (October 1960).

[7] H. Koerner, Electronics **33**, No. 46, 50 (1960).

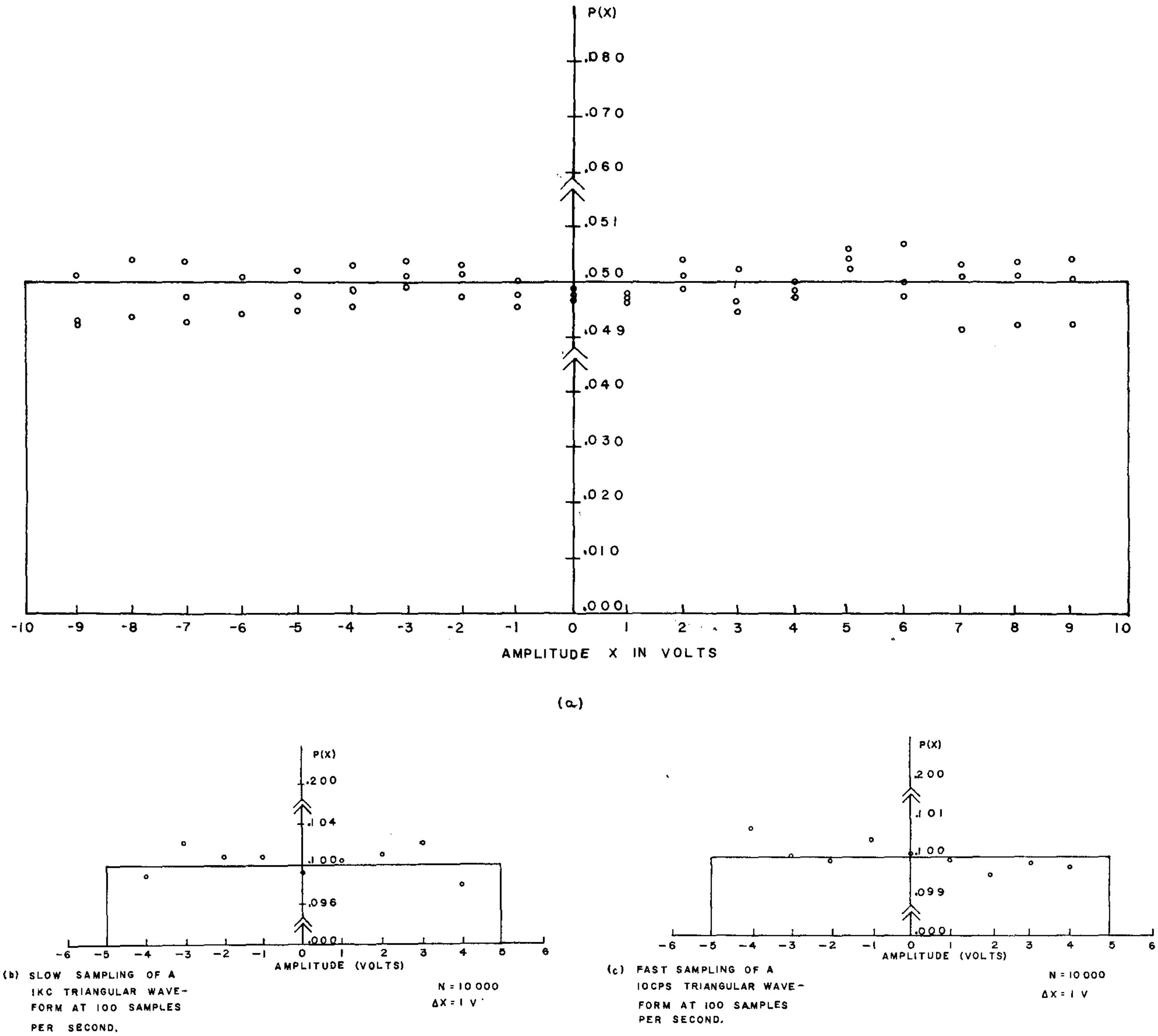


FIG. 6. (a) Probability density estimate of a 20v peak-to-peak 6 cps random phase triangle wave (X = 1 v). (b), (c) Probability density estimates for 10 v peak-to-peak triangular waveforms from a Hewlett-Packard signal generator. Note that the triangle amplitude could be set only with an accuracy of 0.5%, which is less accurate than the analyzer.

$n = 10\,000$ and $\Delta X = 1$ v yielded similar uniformity and repeatability of the estimated probability density; the absolute accuracy was better than that of the 1 kc signal generator used, since the triangle amplitude was known only within 0.5% [Fig. 6(b)].

Figure 7 shows an estimate of the cumulative distribution function

$$\Phi(X_1) = P[x(t) \leq X_1] = \int_{-\infty}^{X_1} \varphi(x)dx$$

for a Gaussian noise generator output. The estimate was obtained through a summation of 24 probability density estimates with $\Delta X = 1$ v and $n = 10\,000$. The sampling rate was 100 samples/sec, which must be regarded as "fast" sampling, since the noise generator output filter cuts off at about 2 cps. This result is shown here only as a test of probability density measurement; errors are seen to accumulate for the larger positive amplitudes. To measure $\Phi(X_1)$ more accurately with the analyzer, one could either set $X + \Delta X/2 = X_1$ and $\Delta X = 50$ v to obtain

$$\Phi(X_1) = P[x(t) \leq X_1] = P[x(t) \leq x + (\Delta X/2)]$$

directly, or one could simply disconnect the precision limiter and $A\Delta X/2$ inputs in Fig. 3(a).

ACKNOWLEDGMENTS

The amplitude distribution analyzer is part of a repetitive analog computer project at The University of Arizona and was started as a term paper project in a graduate course on random processes. The writers are

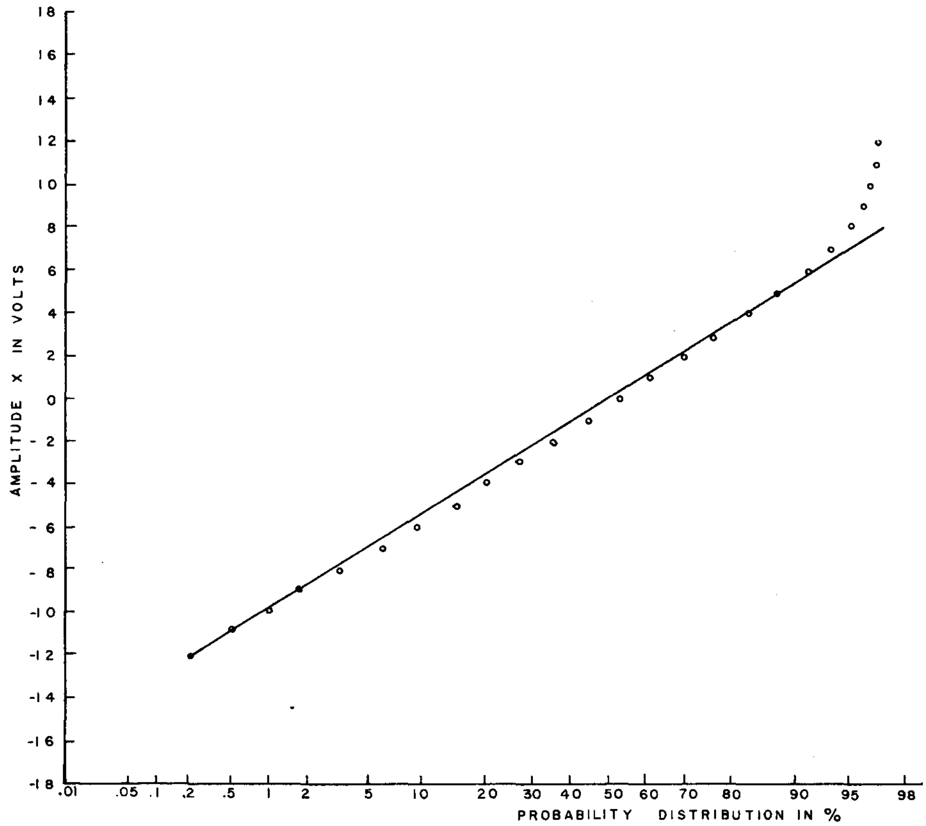


FIG. 7. Test of the accuracy of 24 probability density estimates by error accumulation for Gaussian noise input. Note that this is *not* the way to measure cumulative probabilities.

grateful to the Electrical Engineering Department and to Dr. P. E. Russell, department head, for their continuing support of this project. Thanks are also due James D. Bailey, Richard L. Maybach, and Fred Shaver, all graduate students at the University for their assistance in collecting the data in Figs. 3(b) and 7.

APPENDIX A. VARIANCE OF PROBABILITY ESTIMATES

Even with the most accurate analyzer circuitry, periodic sample probability estimates (3) will exhibit statistical variation due to insufficient sample size and/or statistical dependence of samples. For a stationary random voltage $x(t)$, the variance of the estimate (3) is[8]

$$\mathrm{Var}\{[y]_{\mathrm{av}}\}=\frac{1}{n}\mathrm{Var}\{y\}+\frac{2}{n}\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)E\{y(0)y(k\Delta t)\}-\left(1-\frac{1}{n}\right)(E\{y\})^2,$$

[8] Y. W. Lee, *Statistical Communications Theory* (John Wiley & Sons, Inc., New York, 1960).

which for large sample sizes n reduces to

$$\mathrm{Var}\{[y]_{\mathrm{av}}\}=\frac{2}{n}\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)E\{y(0)y(k\Delta t)\}-(E\{y\})^2,$$

corresponding to statistical errors due to interdependent samples. For small Δt with fixed integration time $T=n\Delta t$, the last expression is approximated by

$$\mathrm{Var}\{\langle y\rangle_T\}=\frac{2}{T}\int_0^T\left(1-\frac{\tau}{T}\right)E\{y(0)y(\tau)\}d\tau-(E\{y\})^2.$$

For estimates (3) of the probability (1) with small ΔX (probability density estimates) one has

$$E\{y\}=\varphi(X)\Delta X,\quad \mathrm{Var}\{y\}=\varphi(1-\varphi\Delta X)\Delta X$$
$$E\{y(0)y(\tau)\}=\varphi_{x_1x_2}(X,\,t;\,X,\,t+\tau)(\Delta X)^2,$$

where $\varphi_{x_1x_2}(X_1,\,t;\,X_2,\,t+\tau)$ is the second order probability density of $x(t)$.[5]