THE ANALOG COMPUTER AS A TEACHING AID IN DIFFENTIAL EQUATIONS (*)

by Ladis D. KOVACH and William COMLEY (**).

SUMMARY.

In order to demonstrate the operation and advantages of the analog computer, a system of differential equations is solved. The equations are derived from a physical model consisting of springs and masses. An analytical solution is also obtained for comparison.

1. *Introduction.*

Students often complete courses in differential equations without ever actually *solving* an equation. True, they obtain expressions which satisfy the given equations, but many times this is done by following a procedure mechanically, without a real understanding of the process. Students need to be shown the relationship between a physical system and the equations describing it. They need to know how the magnitude of the coefficients in the equations affects the solution. Finally, in the study of systems of differential equations, they need to understand how the equations are coupled and what effect the parameters of one equation have on another.

In order to increase understanding in these areas, a series of lectures have been given to the students or differential equations at Pepperdine College during the past two years. These lectures have served not only to strengthen the course but have also acquainted the students with analog computers and computing techniques. In some instances, these lectures have given liberal arts students their only contact with the analog concept.

For maximum effectiveness a logical order is followed in the lecture-demonstration. This order may be conveniently described by the block diagram of Fig. 1.

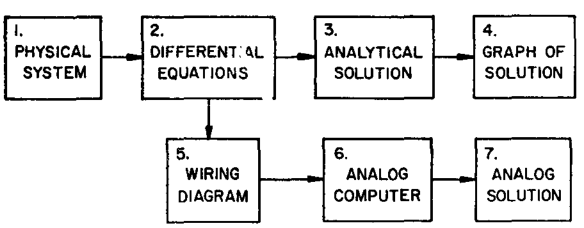

Fig. 1. — Block diagram of lecture-demonstration organization.

2. *Physical System.*

A system consisting of two masses and three springs is convenient to use. Such a two degree-of-freedom system is of sufficient complexity so that all the essential factors may be demonstrated. Later the students are shown how to apply the principles they have learned to more complex systems.

The spring-mass system can be seen in Fig. 2. The upper mass is 200 g, the lower mass is 70 g, the upper spring constant is 74 g/cm, the center spring constant is 58 g/cm, and the lower spring constant is 26 g/cm. These constants produce some interesting motion but, of course, may be changed to suit the material that is available.

Oscillation of the system is produced by displacing the lower mass a distance of 2 cm. This is done several times to give the students a picture of the displacements of the masses and some idea of the coupling that exists.

3. *Differential Equations.*

We proceed to the next subject, that of writing the differential equations of the system. Knowing how to write the differential equations is equally as important as knowing how to solve them. Many otherwise excellent text books neglect this phase of the subject although there are some [1], [2] that pay proper attention to it.

By applying the principle that the equilibrium of the system requires that the sum of the forces on each mass be zero, we obtain equations (1). The parameters are shown in Fig. 3.

For the upper mass :

$$\begin{array}{ccccccc} m_1 \ddot{x}_1 & + & k_1 x_1 & - & k_2 (x_2 - x_1) & = & 0 \\ \text{(inertial force)} & + & \text{(spring force)} & + & \text{(spring force)} & = & 0 \\ & & \text{due to upper} & & \text{due to center} & & \\ & & \text{spring} & & \text{spring} & & \end{array}$$

For the lower mass: (1)

$$\begin{array}{ccccccc} m_2 \ddot{x}_2 & + & k_3 x_2 & + & k_2 (x_2 - x_1) & = & 0 \\ \text{(inertial force)} & + & \text{(spring force)} & + & \text{(spring force)} & = & 0 \\ & & \text{due to lower} & & \text{due to center} & & \\ & & \text{spring} & & \text{spring} & & \end{array}$$

We point out that in writing these equations we have neglected such obvious factors as air resistance and loss of energy in the system due to heating of the springs. Both of these introduce damping in the system whereas equations (1) represent an undamped system. We will have occasion to refer to these facts when we discuss the analog solution.

(*) Manuscrit reçu le 1er octobre 1960.

(**) Douglas Aircraft Co., Inc., El Segundo, California (U.S.A.). The first-named author is also Professor of Mathematics at Pepperdine College, Los Angeles, California.

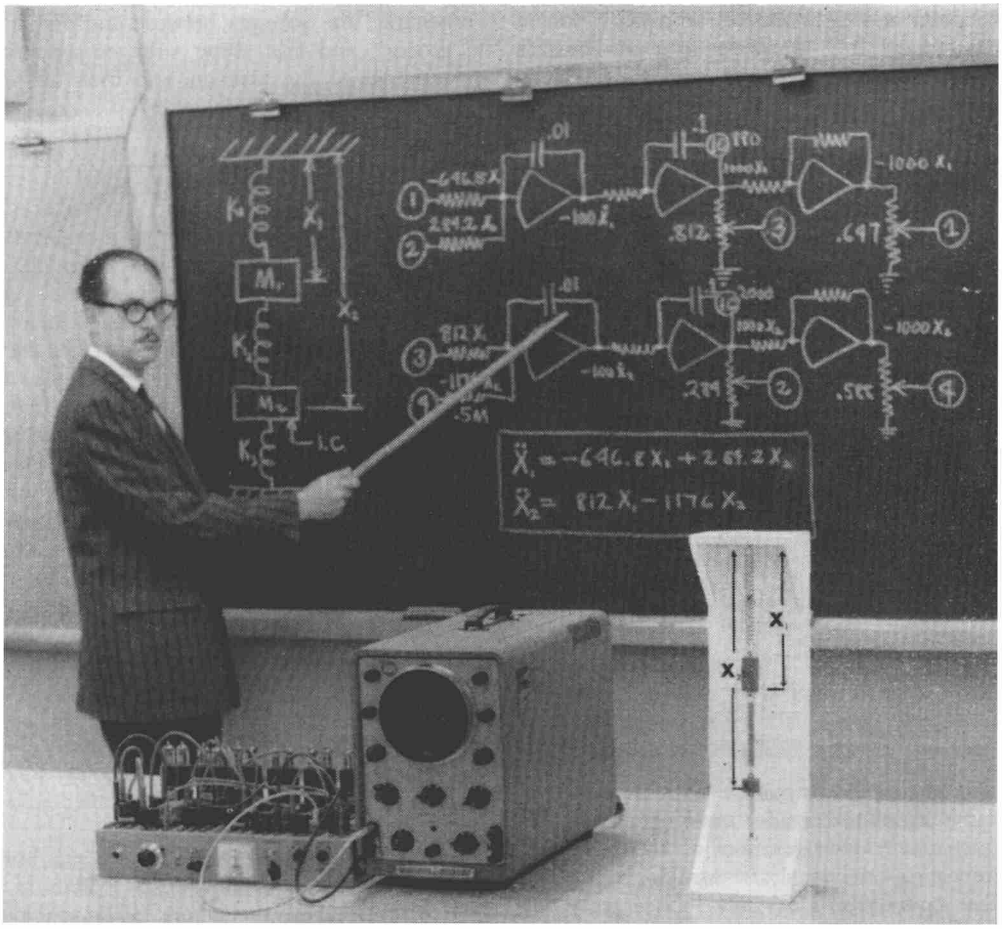

Fig. 2 — Demonstration equipment for solving differential equations on an analog computer.

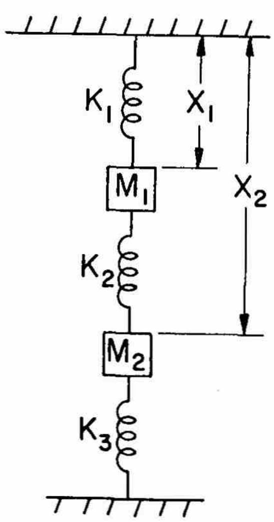

Fig. 3. — Schematic of the physical system.

4. *Analytical Solution.*

In order to use one of the well-known classical methods it is more convenient to rewrite equations (1) in operator and matrix notation obtaining

$$\begin{bmatrix} m_1 p^2 + (k_1 + k_2) & -k_2 \\ -k_2 & m_2 p^2 + (k_2 + k_3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (2)$$

Substituting numerical values for k_1, k_2, k_3, m_1, and m_2 we determine the characteristic equation

$$p^4 + 1823\, p^2 + 529{,}866 = 0 \quad (3)$$

whose roots are $\pm$ 19.1 i, $\pm$ 38.2 i.

Hence the general solution of the system is given by

$$\begin{aligned} x_1 &= c_1 \sin 19.1\, t + c_2 \cos 19.1\, t \\ &\quad + c_3 \sin 38.2\, t + c_4 \cos 38.2\, t \\ x_2 &= c_5 \sin 19.1\, t + c_6 \cos 19.1\, t \\ &\quad + c_7 \sin 38.2\, t + c_8 \cos 38.2\, t \end{aligned}$$

It remains now to determine the constants. We have the initial conditions $x_1(0) = 0.88$* and $x_2(0) = 2$ and also $\dot{x}_1(0) = \dot{x}_2(0) = 0$. To find the constants involves satisfying the initial conditions and also substituting the general solutions (4) into the original equations (1). This results in eight simultaneous linear equations for the eight constants. Solving these we have, finally,

$$\begin{aligned} x_1 &= 1.17 \cos 19.1\, t - 0.29 \cos 38.2\, t \\ x_2 &= 1.17 \cos 19.1\, t + 0.83 \cos 38.2\, t \end{aligned} \quad (5)$$

(*) Note that if x_2 is 2 cm initially, then x_1 will be $\dfrac{2 \times 58}{74 + 58} = 0.88$ cm.

This is the point at which students are usually forced to stop due to lack of time. However, it is not possible for any but an experienced person to visualize the time-histories of the displacements simply by looking at the expressions in (5).

5. *Graph of Solution.*

In order to show how the displacements, x_1 and x_2, vary with time it is necessary to graph equations (5). The student easily recognizes that this can be a tedious, time-consuming task. To illustrate the procedure a few values are assigned to t and the corresponding points are plotted. These can be exhibited in the form of a table as follows :

t	x_2
0	2.00
.025	1.52
.050	0.40
.075	— 0.63
.100	— 1.04
.125	— 0.81
.150	— 0.42
.175	— 0.38
.200	— 0.74

The enormity of this task is made even more apparent when the class is asked to consider some change in one of the parameters. A consideration of the labor involved in repeating the analysis usually results in motivating the students so that they look eagerly for a simpler method. Such a method is made possible by using the analog computer.

6. *Wiring Diagram.*

By substituting numerical values in equations (1) and solving for the highest derivatives, we obtain

$$\ddot{x}_1 = -646.8\, x_1 + 284.2\, x_2$$
$$\ddot{x}_2 = 812\, x_1 - 1176\, x_2 \qquad (6)$$

The wiring diagram of the analog computer can be developed from these equations.

It is desirable, however, to explain the two kinds of operational amplifiers first. At this stage it is not necessary to go into detail regarding such things as open-loop gain, input impedance, drift, etc. The amplifier is introduced simply as a 'black box' which modifies the input in a certain manner to produce an output. A knowledge of the transfer function is sufficient. Fig. 4 shows the necessary information the student needs at this point.

Attention is called to the fact that while three inputs are shown there may be more or less than this number. If we restrict the summer to one input, then we usually call it a 'sign-changer'. The change in sign is also pointed out, as is the fact that the input resistors, R_1, R_2, R_3, can be used to multiply the inputs by a constant. Finally, it is important to explain that all voltages indicated in Fig. 4, as well as in analog circuits in general, are voltages between the points indicated and ground, and that these voltages are *analogous* to the variables of the problem and their derivatives.

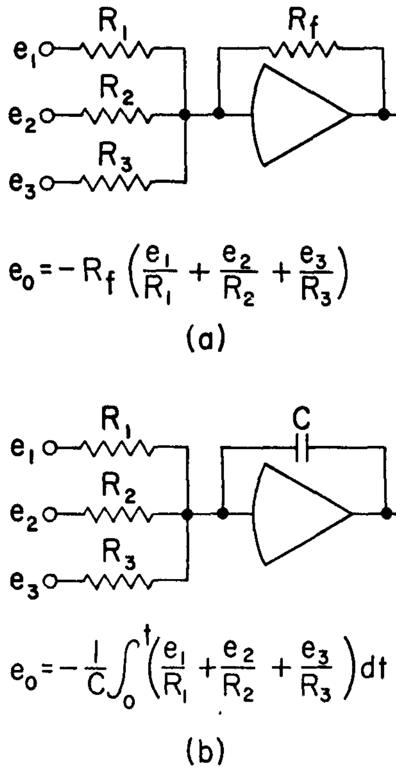

Fig. 4. — Operational amplifiers and their transfer functions.

a) Summer,
b) Integrator.

With this preliminary explanation it is possible to develop the analog wiring diagram for solving equations (6). This is shown in Fig. 5 where the dotted lines indicate those lines that are drawn in *last.*

The diagram emphasizes the fact that differential equations are solved by successively reducing their order until the variables themselves are reached. This process of reduction is called 'integration'.

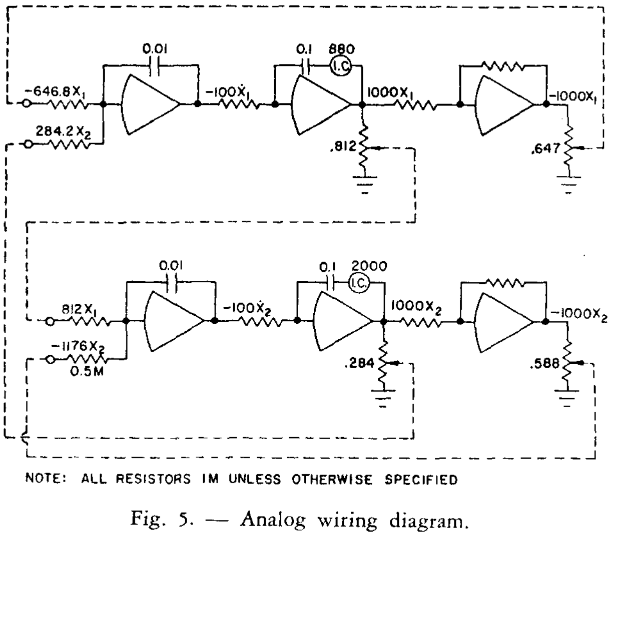

Fig. 5. — Analog wiring diagram.

7. *Analog Computer.*

A small analog computer consisting of six operational amplifiers is used to solve the system. Such a computer is sufficient to demonstrate the principles and has the further advantages of low-cost and portability. Reliability is achieved by using commercially available plug-in operational amplifiers.

The computer shown in Fig. 2 has overall dimensions of 43 cm $\times$ 25 cm $\times$ 18 cm and weighs approximately 6.6 kg. The power supply is on a separate chassis. Six potentiometers are provided for setting the coefficients and a 0.1% linearity potentiometer assures that the others are accurately set under load. The meter serves the double purpose of showing whether the amplifiers are balanced and also to indicate when the coefficient potentiometers are correct.

Two mercury cells are mounted in the computer to provide voltages for initial conditions. Input and output terminals of the amplifiers are arranged so that the proper values of resistance and capacitance may be inserted. Connections between the amplifiers are made with short leads.

8. *Analog Solution.*

The most effective way to present the analog solution is on a cathode-ray oscilloscope as shown in Fig. 2. In this way time-histories of the two velocities can be presented.

By increasing the complexity — and the cost — of the computer slightly it is possible to have a repetitive mode of operation. This mode has the advantage of presenting a stationary solution so that the effect of changing the circuit parameters can be discerned immediately. The importance of this cannot be overemphasized because it is one of the advantages of the analog computer.

The fact that the solution is damped is pointed out to the students. It is explained that the damping is due to the parasitic resistance in the circuit — the resistance of the leads, connections, etc. Because of the slight amount of damping present in the physical system, which was ignored in the equations, the analog solution is a *more realistic* solution than the analytical one. This usually impresses the students since it is an example of how a seemingly undesirable effect (resistance) can be used to advantage.

9. *Conclusion.*

At a modest cost it is possible to build the equipment necessary to present an effective lecture-demonstration on the solution of systems of differential equations. The outline presented has been especially successful because it compares a) the solution on an analog computer with the analytical one, and b) the physical system with the mathematical equations.

BIBLIOGRAPHY

[1] AGNEW, Ralph P. : Differential Equations. McGraw-Hill Book Co., New York, N.Y., 2nd edition, 1960.
An excellent book that covers all phases of differential equations and maintains a good balance between the theoretical and applied.

[2] SPIEGEL, Murray R. : Applied Differential Equations. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1958.
This book contains an unusually large variety of applications and stresses the formulation of the corresponding differential equations.

[3] WARFIELD, John N. : Introduction to Electronic Analog Computers. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1959.
This is an elementary book on analog computers and includes such practical considerations as scaling and checking.